ELSEVIER
Topology and its Applications 96 (1999) 97–108
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

Contractible open 3-manifolds with free covering translation groups ☆

Robert Myers [1]

Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA

Received 8 January 1997; received in revised form 29 December 1997

Abstract

This paper concerns the class of contractible open 3-manifolds which are "locally finite strong end sums" of eventually end-irreducible Whitehead manifolds. It is shown that whenever a 3-manifold in this class is a covering space of another 3-manifold the group of covering translations must be a free group. It follows that such a 3-manifold cannot cover a closed 3-manifold. For each countable free group a specific uncountable family of irreducible open 3-manifolds is constructed whose fundamental groups are isomorphic to the given group and whose universal covering spaces are in this class and are pairwise non-homeomorphic.

Keywords: 3-manifold; Contractible open 3-manifold; Whitehead manifold; Covering space

AMS classification: Primary 57M10, Secondary 57N10; 57M60

1. Introduction

Suppose M is a closed, connected, orientable, irreducible 3-manifold such that $\pi_1(M)$ is infinite. The "universal covering conjecture" states that the universal covering space $\widetilde{M}$ of M must be homeomorphic to $\mathbb{R}^3$. It is known that $\widetilde{M}$ is an irreducible, contractible, open 3-manifold [12]. A *Whitehead manifold* is an irreducible, contractible, open 3-manifold which is not homeomorphic to $\mathbb{R}^3$. The universal covering conjecture is equivalent to the statement that Whitehead manifolds cannot cover closed 3-manifolds. In [15] the author proved that "genus one" Whitehead manifolds cannot non-trivially cover other 3-manifolds, even non-compact ones. Wright [26] extended this result to the much larger class of "eventually end-irreducible" Whitehead manifolds, a class which includes all those Whitehead manifolds which are monotone unions of cubes with a bounded number of

☆ Research at MSRI is supported in part by NSF grant DMS-9022140.
[1] E-mail: myersr@math.okstate.edu.

0166-8641/99/$ – see front matter
PII: S0166-8641(98)00047-9

handles. Tinsley and Wright [22] gave specific examples of Whitehead manifolds which are not eventually end-irreducible and cannot non-trivially cover any other 3-manifolds. They also constructed an uncountable family of Whitehead manifolds which are infinite cyclic covering spaces of other 3-manifolds and deduced from the countability of the set of homeomorphism types of closed 3-manifolds that there must be uncountably many of these which cannot cover closed 3-manifolds; however their methods did not establish which ones these were. In [19] the author constructed a different uncountable family of Whitehead manifolds which are infinite cyclic covering spaces of other 3-manifolds and used different techniques to prove that none of them covers a closed 3-manifold.

This paper combines the methods of [19,26,22] to give a much larger class than in [19] of specific Whitehead manifolds which do not cover closed 3-manifolds but may non-trivially cover other non-compact 3-manifolds, namely the class of "strong end sums along a locally finite tree" of eventually end-irreducible Whitehead manifolds. In fact it is shown that whenever such a manifold covers a 3-manifold the group of covering translations must be a free group (Theorem 3.1). Moreover for any countable free group there are uncountably many specific examples of orientable, irreducible open 3-manifolds whose fundamental groups are isomorphic to the given group and whose universal covering spaces belong to this class and are pairwise non-homeomorphic (Theorem 4.1). There are also uncountably many specific examples in this class which can be only infinite cyclic covering spaces of 3-manifolds and uncountably many specific examples which cannot non-trivially cover any 3-manifold.

The results of [19] use a theorem of Geoghegan and Mihalik [6] which implies that whenever a Whitehead manifold W covers an orientable 3-manifold the group of covering translations must inject into the mapping class group of W. If W covers a closed, orientable, irreducible 3-manifold then the group of covering translations must be finitely generated and torsion-free. In [19] the examples were constructed so that every finitely generated, torsion-free subgroup of their mapping class groups must have a subgroup of finite index which either has infinite abelianization or a non-trivial normal Abelian subgroup. Results of Waldhausen [23], Hass, Rubinstein and Scott [7], Mess [13], Casson–Jungreis [1], and Gabai [5] were then quoted which imply that a closed, orientable, irreducible 3-manifold with such a fundamental group must have universal covering space homeomorphic to $\mathbb{R}^3$.

The present paper avoids the use of the Geoghegan and Mihalik result and the requisite analysis of the mapping class group. For the class of Whitehead manifolds under consideration results of [18] are used to show that the group of covering translations acts on a certain simplicial tree. The Orbit Lemma of [26] and the Special Ratchet Lemma of [22] are then used to prove that this action fixes no vertices, from which the result follows. We remark that the methods by which Tinsley and Wright apply these lemmas in their proof of Theorem 5.3 of [22] could be adapted to prove this fact. However, we present a different, somewhat more direct argument which is closer in spirit to Wright's proof of the Main Theorem of [26]. We also give an alternative, somewhat shorter proof of the special case of the Orbit Lemma that we use.

The Whitehead manifolds considered in [22,19], and this paper are all "end sums" of Whitehead manifolds; they are obtained by gluing together a collection of Whitehead manifolds in a certain way (see the next section for the precise definition). The summands in [22] are members of a certain uncountable collection of genus one Whitehead manifolds discovered by McMillan [11]; the summands in [19] are members of a different uncountable collection of genus one Whitehead manifolds chosen so that the mapping class group of the end sum will have the appropriate structure as described above. However, the main difference is not in the summands, but in how they are glued together. The examples of [19] are all "strong" end sums which have a certain "rigidity up to isotopy" in their construction. The end sums in [22] are not strong end sums; in fact it follows from Proposition 2.1 below that these manifolds cannot be expressed in any way as strong end sums, even though by Proposition 2.2 below their summands can be glued together in a different fashion to obtain different manifolds which are strong end sums. Thus the results of this paper apply to all the examples of [19] but none of the examples of [22]. The question of which of them cannot cover closed 3-manifolds (conjecturally all of them) is still open.

2. Background material

For general background on 3-manifold topology see [8,9]. We denote the manifold theoretic boundary and interior of a manifold M by ∂M and $int\, M$, respectively. We denote the topological boundary, interior, and closure of a submanifold M of a manifold N by $Fr_N(M)$, $Int_N(M)$, and $Cl_N(M)$, respectively, with the subscript deleted when N is clear from the context. The *exterior* of M in N is the closure of the complement of a regular neighborhood of M in N. M is *open* if $\partial M = \emptyset$ and no component of M is compact. A continuous map $f : M \to N$ of manifolds is ∂-*proper* if $f^{-1}(\partial N) = \partial M$. It is *end-proper* if preimages of compact sets are compact. It is *proper* if it has both these properties. These terms are applied to a submanifold if its inclusion map has the corresponding property. Two codimension one submanifolds M_0 and M_1 of N, each of which is either proper in N or is a submanifold of ∂N, are *parallel* if some component of $N - (M_0 \cup M_1)$ has closure homeomorphic to $M_0 \times [0,1]$ with $M_i = M_0 \times \{i\}$, $i = 0, 1$. A proper codimension one submanifold of N is ∂-*parallel* if it is parallel to a submanifold of ∂N.

An *exhaustion* $\{K_n\}_{n \geqslant 0}$ for a connected, non-compact manifold W is a sequence of compact, connected, codimension zero submanifolds of W whose union is W, such that $K_n \subseteq Int\, K_{n+1}$, $K_n \cap \partial W$ is either empty or a codimension zero submanifold of ∂W, and $W - Int\, K_n$ has no compact components.

A connected, non-compact 3-manifold W is *eventually end-irreducible* if it has an exhaustion $\{K_n\}$ such that $Fr\, K_n$ is incompressible in $W - Int\, K_0$ for all $n \geqslant 0$. We also say that W is *end-irreducible* rel K_0. W is *eventually* π_1-*injective at* ∞ if there is a compact subset J of W such that for every compact subset K of W containing J there is a compact subset L of W containing K such that every loop in $W - L$ which is null-homotopic in $W - J$ is null-homotopic in $W - K$. We also say that W is π_1-*injective at* ∞ rel J. It is a standard exercise to show that W is eventually end-irreducible if and only if it is eventually π_1-injective at ∞. Note in particular that if W is end-irreducible rel K_0, then it is π_1-injective at ∞ rel K_0.

Let V be an irreducible non-compact 3-manifold such that either $\partial V = \emptyset$ or each component of ∂V is a plane. A proper plane P in V is *trivial* if some component of $V - P$ has closure homeomorphic to $\mathbb{R}^2 \times [0, \infty)$ with $P = \mathbb{R}^2 \times \{0\}$. V is $\mathbb{R}^2$-*irreducible* every proper plane in V is trivial (hence $\partial V = \emptyset$ or $V = \mathbb{R}^2 \times [0, \infty)$); it is *aplanar* if every proper plane in V is either trivial or ∂-parallel. A *partial plane* is a simply connected, non-compact 2-manifold with non-empty boundary. V is *strongly aplanar* if it is aplanar and given any proper 2-manifold $\mathcal{P}$ in V each component of which is a partial plane, there is a collar on ∂V which contains $\mathcal{P}$. V is *anannular at* ∞ if for every compact subset K of V there is a compact subset L of V containing K such that $V - L$ is *anannular*, i.e., every proper, incompressible annulus in $V - L$ is ∂-parallel.

Now suppose we are given a countable simplicial tree Γ to each vertex v_i of which we have associated a connected, oriented, irreducible, non-compact 3-manifold V_i whose boundary is a non-empty disjoint union of planes. Suppose that to each edge e_k of Γ we have associated a component of ∂V_i and a component of ∂V_j, where e_k has endpoints v_i and v_j and no boundary plane is associated to different edges. The connected, oriented, non-compact 3-manifold W obtained by gluing each such pair of planes by an orientation reversing homeomorphism is called the *plane sum* of the V_i along Γ. The image in W of the pair of planes identified as above is denoted by E_k and is called a *summing plane*. The plane sum is *degenerate* if either some summing plane is trivial or ∂-parallel in W or two distinct summing planes are parallel in W. Theorem 3.2 of [18] gives necessary and sufficient conditions on Γ and the V_i for the plane sum to be non-degenerate. For our present purposes Corollary 3.3 of [18], which states that the plane sum is non-degenerate if no summand V_i has a boundary plane E_k such that $E_k \cup \mathit{int}\, V_i$ is homeomorphic to $\mathbb{R}^2 \times [0, \infty)$, will suffice because in our case $\mathit{int}\, V_i$ will be a Whitehead manifold. The plane sum is *strong* if it is non-degenerate and each summand is strongly aplanar and anannular at ∞.

Proposition 2.1. *Let W be a non-degenerate plane sum of aplanar* 3*-manifolds along a locally finite tree. Let W' be a strong plane sum. Let $\mathcal{E}$ and $\mathcal{E}'$ be the unions of the respective sets of summing planes. Suppose $g : W \to W'$ is a homeomorphism. Then g is ambient isotopic* rel ∂W *to a homeomorphism h such that $h(\mathcal{E}) = \mathcal{E}'$.*

Proof. This is Theorem 4.3 of [18]. For the sake of completeness we briefly sketch the ideas involved in the proof. By Proposition 2.1 of [17] a set of simple closed curve intersections of $g(\mathcal{E})$ and $\mathcal{E}'$ can be removed by an ambient isotopy provided that there is no infinite nesting among its elements on some component of $g(\mathcal{E})$ or $\mathcal{E}'$. Non-degeneracy and anannularity at ∞ insure either that there is no such infinite nesting or that it can be removed by an ambient isotopy. Strong aplanarity is then used to isotop $g(\mathcal{E})$ off $\mathcal{E}'$. Finally one uses aplanarity and non-degeneracy to isotop $g(\mathcal{E})$ to $\mathcal{E}'$. $\square$

Now suppose that given Γ we have associated to each vertex v_i a connected, open, irreducible, oriented 3-manifold W_i, and that to each edge e_k we have associated an end-proper ray (a space homeomorphic to $[0,\infty)$) in W_i and an end-proper ray in W_j, where e_k has endpoints v_i and v_j, the rays associated to different edges are disjoint and their union is end-proper. The exterior V_i of the union of the rays contained in W_i is then bounded by planes. Note that *int* V_i is homeomorphic to W_i. The plane sum W of the V_i along Γ is called an *end sum* of the W_i along Γ. (Note that W depends on the choice of the rays; this dependence is investigated further in [18].) A *strong end sum* is one whose associated plane sum is strong.

We conclude this section with some remarks about the existence of strong end sums. In the present context the following is the most relevant fact; more general results may be found in [17,18].

Proposition 2.2. *Given a countable, locally finite tree Γ, a collection $\{W_i\}$ of connected, irreducible, oriented, one-ended open* 3*-manifolds, and a bijection between the vertices of Γ and $\{W_i\}$, there exists a strong end sum of the W_i along Γ.*

Proof. This is a special case of Theorem 5.1 of [18]. □

For later reference we briefly describe the construction of the rays required in the proof of this result. Suppose V is a connected, orientable, irreducible, one-ended, non-compact 3-manifold whose boundary is either empty or consists of a finite set of disjoint planes. An exhaustion $\{C_n\}$ for V is *nice* if for all $n \geqslant 1$ one has that $C_n - \mathit{Int}\, C_{n-1}$ is irreducible, ∂-irreducible, and anannular, and that for all $n \geqslant 0$ one has that each component of $\mathit{Fr}\, C_n$ has positive genus and negative Euler characteristic, and if $\partial V \neq \emptyset$, one has that $C_n \cap \partial V$ consists of a single disk in each component of ∂V. One says that V is *nice* if it has a nice exhaustion.

Proposition 2.3. *If V is nice, then V is strongly aplanar and anannular at ∞.*

Proof. This follows from Theorem 5.3 and Lemma 1.3(6) of [17]. □

Given W_i one chooses an exhaustion $\{K_n\}$ for W_i with each ∂K_n connected and of positive genus. If ν rays are required, then for each $n \geqslant 1$ one chooses a disjoint union of ν proper arcs in $K_n - \mathit{Int}\, K_{n-1}$ each component of which joins $\mathit{Fr}\, K_{n-1}$ to $\mathit{Fr}\, K_n$. This is done so that the endpoints match up on $\mathit{Fr}\, K_n$ so as to give ν rays in W_i. Then we obtain an exhaustion $\{C_n\}$ for V_i by letting $C_0 = K_0$ and for $n \geqslant 1$ letting C_n be the union of K_0 and the exterior in $K_n - \mathit{Int}\, K_0$ of its intersection with these rays. All that remains is to note that by Theorem 1.1 of [16] one can choose the arcs so that $C_n - \mathit{Int}\, C_{n-1}$ is irreducible, ∂-irreducible, and anannular.

In Section 4 we will give explicit constructions of examples of this type which do not rely on Theorem 1.1 of [16].

3. The general result

Theorem 3.1. *Let W be a strong end sum of eventually end-irreducible Whitehead manifolds W_i along a locally finite tree Γ. If W is a covering space of a 3-manifold M, then there is a simplicial action of $\pi_1(M)$ on Γ under which no non-trivial element of $\pi_1(M)$ fixes a vertex of Γ. Hence*

(1) *$\pi_1(M)$ is a free group.*
(2) *M cannot be a closed 3-manifold.*
(3) *If Γ has countably many ends, then $\pi_1(M)$ is cyclic.*
(4) *If the number of ends of Γ is finite and greater than two, then $\pi_1(M)$ is trivial, i.e., $M = W$.*

Proof. We first show how to deduce (1)–(4) from the main statement of the theorem.

(1) $\pi_1(M)$ has a subgroup H of index at most two which acts on Γ without inversions of the edges, hence acts freely on Γ, hence is free. It follows that $\pi_1(M)$ is itself free [21].

(2) If M were closed then it would be a connected sum of 2-sphere bundles over S^1 [8, Theorem 5.2], hence would not be aspherical, hence its universal covering space would not be contractible.

(3) If $rank\, \pi_1(M) \geqslant 2$, then Γ has uncountably many ends.

(4) Suppose A is an axis for the action of $\pi_1(M)$ on Γ, i.e., A is a subtree isomorphic to a triangulation of $\mathbb{R}$ which is invariant under the infinite cyclic action (see [20]). Since Γ has at least three ends some component of $\Gamma - A$ has non-compact closure T, and the translates of T yield infinitely many ends of Γ.

We now prove the main statement of the theorem. Let $G \cong \pi_1(M)$ be the group of covering translations. By Proposition 2.1 each $g \in G$ is isotopic to a homeomorphism h such that $h(\mathcal{E}) = \mathcal{E}$, where $\mathcal{E}$ is the union of the summing planes of W. Thus h determines an element of $Aut(\Gamma)$. We claim that this element depends only on g. We repeat the argument of Theorem 3.2 of [19]. If h' were a homeomorphism isotopic to g which determined a different automorphism, then h and h' would send some summing plane E_i to different summing planes E_j and E_k, hence they would be ambient isotopic in W. But by Theorem 5 of [25] disjoint, ambient isotopic, non-trivial, proper planes in an irreducible 3-manifold must be parallel. This contradicts the non-degeneracy of strong end sums.

Thus we have a well defined action of G on Γ. We next state the results of [26,22] that we shall need in order to prove that no vertex is fixed by a non-trivial element of G.

Let G be a group acting on an n-manifold W. One says that G acts *without fixed points* if the only element of G fixing a point is the identity. G acts *totally discontinuously* if for every compact subset C of W one has that $g(C) \cap C = \emptyset$ for all but finitely many elements of G. (In [26] the term "properly discontinuously" is used for this property; we follow Freedman and Skora's terminology [4] in order to avoid confusion with other meanings of this term.) Let $p : W \to Y$ be the projection to the orbit space Y of the action. Then G acts without fixed points and totally discontinuously on W if and only if p is a regular covering map with group of covering translations G and Y is an n-manifold. (See [10].) In this case if W is contractible, then G must be torsion-free (see, e.g., [15,26]).

Proposition 3.2 (Orbit Lemma (Wright)). *Let W be a contractible, open n-manifold, $n \geqslant 3$. Let g be a non-trivial homeomorphism of W onto itself such that the group $\langle g \rangle$ of homeomorphisms generated by g acts without fixed points and totally discontinuously on W. Given compact subsets B and Q of W, there is a compact subset C of W containing B such that every loop in $W - C$ is homotopic in $W - B$ to a loop in $W - \bigcup_{i=-\infty}^{\infty} g^i(Q)$.*

Proof. Except for the statement that C contains B this is Lemma 4.1 of [26]; we can clearly enlarge the C of that result to satisfy this requirement.

We now give an alternate proof for the special case in which W is an irreducible 3-manifold. The quotient manifold $Y = W/\langle g \rangle$ is an irreducible open 3-manifold having the homotopy type of a circle. Any irreducible open 3-manifold with locally free fundamental group has an exhaustion by cubes with handles (Theorem 2 of [3]). Let $\{Y_n\}$ be such an exhaustion for Y. We may assume that $\pi_1(Y_0) \to \pi_1(Y)$ is onto and $p(Q) \subseteq \mathit{Int}\, Y_0$, where $p : W \to Y$ is the covering projection. Thus $\bigcup_{i=-\infty}^{\infty} g^i(Q) \subseteq \mathit{Int}\, p^{-1}(Y_0)$. Now $p^{-1}(Y_0)$ is a non-compact cube with handles. There is a finite set of disjoint, proper disks in $p^{-1}(Y_0)$ whose union splits $p^{-1}(Y_0)$ into a compact cube with handles H which contains $B \cap p^{-1}(Y_0)$ and a 3-manifold H' whose components are non-compact cubes with handles. These splitting disks can be chosen disjoint from B. Let $C = B \cup H$. Suppose γ is a loop in $W - C$. Homotop γ so that it is in general position with respect to $\partial H'$. Then it meets H' in a finite set of paths γ_j. Since the components of H' are cubes with handles each γ_j can be homotoped rel $\partial \gamma_j$ to a path γ_j' in $\partial H'$. This can be done so that no γ_j' meets a splitting disk. Thus γ is homotopic in $W - C$, and hence in $W - B$, to a loop γ' which lies in $W - \mathit{Int}\, p^{-1}(Y_0)$ and hence in $W - \bigcup_{i=-\infty}^{\infty} g^i(Q)$. □

Proposition 3.3 (Special Ratchet Lemma (Tinsley–Wright)). *Let W be an open n-manifold and W_0 an open subset of W with closure V_0. Suppose W_0 is π_1-injective at ∞* rel J*, V_0 is an n-manifold, ∂V_0 is proper and bicollared in W, and each component of ∂V_0 is simply connected. Let g be a homeomorphism of W onto itself such that each of $g(J)$ and $g^{-1}(J)$ can be ambiently isotoped into W_0. Then there is a compact subset R of W containing J such that a loop in $W - \bigcup_{i=-\infty}^{\infty} g^i(R)$ is null-homotopic in $W - J$ if and only if it is null-homotopic in $W - g^i(J)$ for each $i \in \mathbb{Z}$.*

Proof. This is a slight variation of Lemma 5.1 of [22] which has the same proof. □

Continuing with the proof of Theorem 3.1, we note that the hypotheses of the Special Ratchet Lemma are clearly satisfied when G acts on Γ with fixed points, i.e., some non-trivial $g \in G$ is isotopic to h such that $h(V_0) = V_0$ for the plane summand V_0 associated to an end summand W_0. We shall prove that W_0 is π_1-*trivial at* ∞, i.e., for every compact subset A of W_0 there is a compact subset A^* of W_0 containing A such that every loop in $W_0 - A^*$ is null-homotopic in $W_0 - A$. By a result of Edwards [2] and Wall [24] every irreducible, contractible, open 3-manifold which is π_1-trivial at ∞ must be homeomorphic to $\mathbb{R}^3$. This contradicts the assumption that W_0 is a Whitehead manifold.

So, let A be a compact subset of W_0. Now W_0 is π_1-injective at ∞ rel J for some compact subset J of W_0. By the Special Ratchet Lemma there is a compact subset R of

W containing J such that a loop in $W - \bigcup_{i=-\infty}^{\infty} g^i(R)$ is null-homotopic in $W - J$ if and only if it is null-homotopic in $W - g^i(J)$ for all $i \in \mathbb{Z}$. Let $N = \partial V_0 \times [0,1]$ be a collar on ∂V_0 in V_0 such that $\partial V_0 \times \{0\} = \partial V_0$ and $N \cap (A \cup J) = \emptyset$. Let $R_0 = R \cap Cl(V_0 - N)$. Then R_0 is a compact subset of W_0 which contains J. Let $K = A \cup R_0$. Since W_0 is π_1-injective at ∞ rel J there is a compact subset L of W_0 containing K such that loops in $W_0 - L$ which are null-homotopic in $W_0 - J$ are null-homotopic in $W_0 - K$. Apply the Orbit Lemma with $B = L$ and $Q = R$ to get a compact subset C of W containing L such that every loop in $W - C$ is homotopic in $W - L$ to a loop in $W - \bigcup_{i=-\infty}^{\infty} g^i(R)$. By enlarging C, if necessary, we may assume that $C \cap N$ consists of cylinders $D_j \times [0,1]$, where D_j is a disk in the component E_j of ∂V_0. There is an $s \in (0,1)$ such that the collar $N_s = \partial V_0 \times [0,s]$ misses L. Let $C_0 = C \cap Cl(V_0 - N_s)$.

We claim that we may take $A^* = C_0$. Consider a loop γ in $W_0 - C_0$. We will show that γ is null-homotopic in $W_0 - A$. First note that $\gamma \cap C$ is contained in the union of the $D_j \times (0,s)$. We can homotop γ in $W_0 - C_0$, if necessary, so that it misses the union of the $\{x_j\} \times [0,s]$, where x_j is a point in the interior of D_j. By pushing radially outward from $\{x_j\} \times [0,s]$ in each $D_j \times [0,s]$ and then off $D_j \times [0,s]$ we obtain a homotopy of γ in $W_0 - C_0$ to a loop γ' in $W_0 - (W_0 \cap C)$. Now γ' is homotopic in $W - L$ to a loop γ'' in $W - \bigcup_{i=-\infty}^{\infty} g^i(R)$. Since W is contractible γ'' is null-homotopic in W. Since $\langle g \rangle$ is totally discontinuous γ'' is null-homotopic in $W - g^i(J)$ for some i. Since γ'' lies in $W - L$ the Special Ratchet Lemma implies that γ'' is null-homotopic in $W - J$. Since $J \subseteq L \subseteq C_0$ we have that γ is null-homotopic in $W - J$. Since γ lies in $W_0 - J$ and the components of ∂V_0 are simply connected we have that γ is null-homotopic in $W_0 - J$. Thus γ is null-homotopic in $W_0 - K \subseteq W_0 - A$, as required. □

4. Specific examples

Theorem 4.1.

(1) *Given any countable free group F there are uncountably many specific irreducible, orientable, open 3-manifolds X such that $\pi_1(X) \cong F$, any 3-manifold M covered by the universal covering space W of X must have free fundamental group, and the W are pairwise non-homeomorphic.*

(2) *If $F \cong \mathbb{Z}$, then X can be chosen so that $\pi_1(M)$ must be infinite cyclic.*

(3) *If F is trivial, then $X = W$ can be chosen so that $M = W$.*

Proof. (1) It suffices to consider the case when F has rank two. The construction will be a generalization of that of Theorem 6.1 of [19]. Fig. 1 shows a six component tangle λ in a 3-ball B called the true lover's 6-tangle. By Proposition 4.1 of [14] the exterior of λ is *excellent*, i.e., it is irreducible, ∂-irreducible, anannular, and atoroidal, contains a proper incompressible surface, and is not a 3-ball. We recall that this is proven by splitting this manifold along a certain collection of incompressible disks with two holes to obtain a set of cubes with handles (see Fig. 3 of [14]) and then analyzing the intersection of a supposed essential surface of the proscribed type with these cubes with handles. It follows

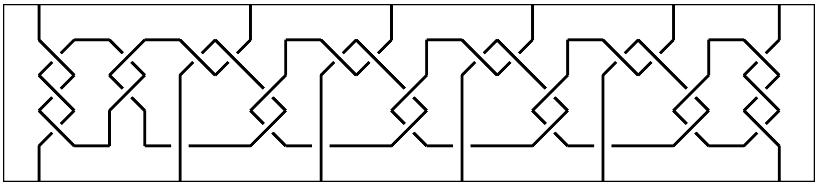

Fig. 1. The 6-tangle λ.

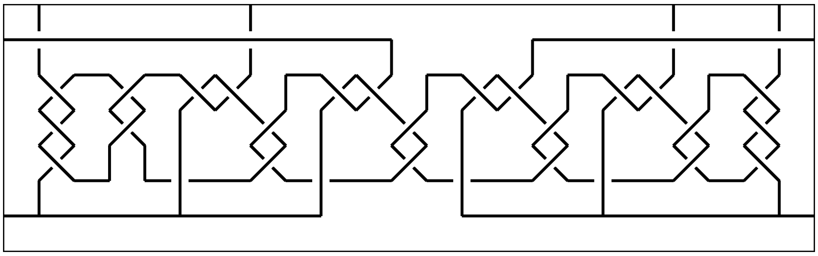

Fig. 2. The graph ξ.

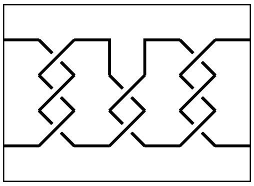

Fig. 3. The 2-tangle μ.

immediately from the proof that each of the k-tangles consisting of $k \geqslant 2$ consecutive components of λ also has excellent exterior. By sliding the endpoints of the arcs of λ one sees that the exterior of λ is homeomorphic to the exterior of the graph ξ in Fig. 2. By deleting the first, second, fifth, and sixth arcs we obtain the 2-tangle μ in Fig. 3, which thus has excellent exterior.

We next identify the disks which are the left and right sides of the rectangular solid B in Figs. 2 and 3 to obtain a solid torus K. This is done so that μ becomes a simple closed curve σ and ξ becomes a graph θ consisting of σ together with four disjoint arcs α^1, α^2, α^3, α^4 joining σ to ∂K. It follows from Lemma 2.1 of [16] that the exteriors of σ and of θ in K are excellent.

Now let L be a regular neighborhood of σ in K. We construct a genus one Whitehead manifold U with exhaustion $\{K_n\}$ by using as models for (K_n, K_{n-1}) the pair (K, L). This is done so that the copies α_n^j of the α^j match up along their endpoints to give end-proper rays ρ_j in U. We then let V be U minus the interior of a regular neighborhood N of the union of these rays. We choose N so that its intersection N_n with $K_n - \mathit{int}\, K_{n-1}$ is a regular neighborhood of the union of the α_n^j. We then let C_n be $Cl(K_n - N_n)$ for $n \geqslant 1$

and $C_0 = K_0$. Since $K_n - \mathit{int}\, K_{n-1}$ and $C_n - \mathit{Int}\, C_{n-1}$ are excellent we have that U is an eventually end-irreducible Whitehead manifold and V is nice.

We now identify the boundary planes of V in pairs to obtain an orientable 3-manifold X with $\pi_1(X)$ free of rank two. The universal covering space W of X is then an end sum of Whitehead manifolds W_i each of which is homeomorphic to U such that the plane summands V_i are homeomorphic to V. We then apply Theorem 3.1.

We next show how to get uncountably many examples of this type with pairwise non-homeomorphic universal covering spaces.

If one changes the sense of the central clasp in the figures by changing the two overcrossings to undercrossings, thereby getting a new σ and θ, then the same arguments show that their exteriors in K are excellent. Denote the old and new versions by the subscripts 0 and 1, respectively. Embed K in S^3 in a standard way so that a line segment running along the bottom front edge of B becomes a simple closed curve ℓ in ∂K which bounds a disk in $S^3 - \mathit{int}\, K$. Then σ_0 and σ_1 become the knots 8_5 and 8_{19} in S^3 with normalized Alexander polynomials $5 - 4(t + t^{-1}) + 3(t^2 + t^{-2}) - (t^3 + t^{-3})$ and $1 - (t^2 + t^{-2}) + (t^3 + t^{-3})$, respectively. It then follows that there is no homeomorphism from the exterior of σ_0 in K to that of σ_1 in K which carries ℓ to a curve homologous to $\pm\ell$, since if there were, then one could extend it to a homeomorphism of the exteriors in S^3 of these two knots.

Let $s = \{s_n\}_{n\geqslant 1}$ be an infinite sequence of 0's and 1's. Carry out the construction as before by modeling the pair (K_n, K_{n-1}), for $n \geqslant 1$, on (K, L_i), where L_i is a regular neighborhood of σ_i in K and $i = s_n$. Do this so that the copy ℓ_n of ℓ in ∂K_n is null-homologous in $K_{n+1} - \mathit{int}\, K_n$. (Up to orientation and isotopy there is a unique such curve.)

Label the various manifolds arising in the construction associated to s by a superscript s. If $f : U^s \to U^t$ is a homeomorphism, then Lemma 3.3 of [15] implies that f can be isotoped so that for some a and b one has $f(K^s_{a+m}) = K^t_{b+m}$ for all $m \geqslant 0$. Thus $s_{a+m} = t_{b+m}$ for all $m \geqslant 0$.

One could now note that this last equation generates an equivalence relation on the set $\{0, 1\}^\omega$ of all such sequences and that there are uncountably many equivalence classes. In keeping with the desire to make our examples as explicit as possible, however, we prefer a more concrete approach which exhibits an explicit subset $\mathcal{S}$ of $\{0, 1\}^\omega$ for which the corresponding Whitehead manifolds are non-homeomorphic. We define $\mathcal{S}$ and define a bijection $\varphi : \{0, 1\}^\omega \to \mathcal{S}$ as follows. Let $x \in \{0, 1\}^\omega$. Then $s = \varphi(x)$ will consist of strings of consecutive 0's which are separated by single 1's. The length of the nth string of 0's is $d_n = r_1 r_2 \cdots r_n$, where $r_j = 3^{(2^{j-1})}$ if $x_j = 0$ and $r_j = 5^{(2^{j-1})}$ if $x_j = 1$. Thus $d_n = 3^u 5^v$, where the total exponent sum $u + v = 1 + 2 + 4 + 8 + \cdots + 2^{n-1} = 2^n - 1$.

Suppose $t = \varphi(y)$ is another sequence such that for some a and b one has $s_{a+m} = t_{b+m}$ for all $m > 0$. Locate the first 1 in this common tail. It is followed by a string of $3^u 5^v$ 0's for some unique u and v. Then $u + v = 2^n - 1$ for a unique n, and so this is the nth string of 0's in both s and t. Note that $n > 1$. Suppose $d_n = r_1 r_2 \cdots r_n = q_1 q_2 \cdots q_n$ where the r_j and q_j correspond to the x_j and y_j as above. Then $d_{n-1} = d_n / r_n$; let $p_{n-1} = d_n / q_n$. If $r_n = 3^{(2^{n-1})}$, then since p_{n-1} has exponent sum in 3 at most $2^{n-1} - 1$ we must have $q_n = 3^{(2^{n-1})}$; since a similar argument holds for powers of 5 we have that $r_n = q_n$. We inductively conclude that $r_j = q_j$, and hence $x_j = y_j$, for $1 \leqslant j \leqslant n$. Applying this argument to all $n' > n$ we get that $x = y$ and $s = t$.

Thus we have uncountably many non-homeomorphic genus one Whitehead manifolds U^s. We construct the corresponding V^s, X^s, and W^s. The W_i^s are all homeomorphic to U^s. It then follows from Proposition 2.1 that if W^s and W^t are homeomorphic, so are U^s and U^t, hence $s = t$.

(2) We perform the analogous construction with the first and last arcs deleted. See Theorem 6.1 of [19].

(3) One can carry out the construction of V as above with any finite number ν of boundary planes by using the true lover's $\nu + 2$-tangle. Thus given any locally finite tree Γ one can construct the corresponding strong end sum. One can then choose Γ to have the wrong number of ends or, for variety, let Γ be arbitrary but choose one W_0 which is not homeomorphic to any of the other W_i, thereby creating a fixed vertex for the action on Γ. $\square$

References

[1] A. Casson and D. Jungreis, Convergence groups and Seifert fibered 3-manifolds, Invent. Math. 118 (1994) 441–456.
[2] C.H. Edwards, Open 3-manifolds which are simply connected at infinity, Proc. Amer. Math. Soc. 14 (1963) 391–395.
[3] B. Freedman and M. Freedman, Kneser–Haken finiteness for bounded 3-manifolds, locally free groups, and cyclic covers, Topology 37 (1998) 133–147.
[4] M. Freedman and R. Skora, Strange actions of groups on spheres, J. Differential Geom. 25 (1987) 75–98.
[5] D. Gabai, Convergence groups are Fuchsian groups, Ann. of Math. 136 (1992) 447–510.
[6] R. Geoghegan and M. Mihalik, The fundamental group at infinity, Topology 35 (1996) 655–669.
[7] J. Hass, H. Rubinstein and P. Scott, Compactifying coverings of closed 3-manifolds, J. Differential Geom. 30 (1989) 817–832.
[8] J. Hempel, 3-Manifolds, Ann. of Math. Stud. 86 (Princeton, 1976).
[9] W. Jaco, Lectures on three-manifold topology, CBMS Regional Conference Series in Math. 43 (Amer. Math. Soc., 1980).
[10] W. Massey, Algebraic Topology: An Introduction, Graduate Texts in Math. 56 (Springer, Berlin, 1977).
[11] D.R. McMillan Jr, Some contractible open 3-manifolds, Trans. Amer. Math. Soc. 102 (1962) 373–382.
[12] W. Meeks, L. Simon and S.T. Yau, Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature, Ann. of Math. 116 (1982) 621–659.
[13] G. Mess, Centers of 3-manifold groups and groups which are coarse quasi-isometric to planes, Preprint.
[14] R. Myers, Homology cobordisms, link concordances, and hyperbolic 3-manifolds, Trans. Amer. Math. Soc. 278 (1983) 271–288.
[15] R. Myers, Contractible open 3-manifolds which are not covering spaces, Topology 27 (1988) 27–35.
[16] R. Myers, Excellent 1-manifolds in compact 3-manifolds, Topology Appl. 49 (1993) 115–127.
[17] R. Myers, Attaching boundary planes to irreducible open 3-manifolds, Quart. J. Math. Oxford Ser. (2) 48 (1997) 363–404.

[18] R. Myers, End sums of irreducible open 3-manifolds, Quart. J. Math. Oxford Ser. (2), to appear.
[19] R. Myers, Contractible open 3-manifolds which non-trivially cover only non-compact 3-manifolds, Topology 38 (1999) 85–94.
[20] P. Shalen, Dendrology of groups: an introduction, in: Essays in Group Theory, Math. Sci. Res. Inst. Publ. 8 (Springer, New York, Berlin, 1987) 265–319.
[21] J. Stallings, On torsion free groups with infinitely many ends, Ann. of Math. 88 (1968) 312–334.
[22] F. Tinsley and D. Wright, Some contractible open manifolds and coverings of manifolds in dimension three, Topology Appl. 77 (1997) 291–301.
[23] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, Ann. of Math. 87 (1968) 56–88.
[24] C.T.C. Wall, Open 3-manifolds which are 1-connected at infinity, Quart. J. Math. Oxford Ser. (2) 16 (1965) 263–268.
[25] B. Winters, Properly homotopic nontrivial planes are parallel, Topology Appl. 48 (1992) 235–243.
[26] D. Wright, Contractible open manifolds which are not covering spaces, Topology 31 (1992) 281–291.